12 Convex Quadrilaterals

Definition (quadrilateral)
Let $\{A, B, C, D\}$ be a set of four points in a metric geometry no three of which are collinear. If no two of $\text{int}(\overline{AB})$, $\text{int}(\overline{BC})$, $\text{int}(\overline{CD})$ and $\text{int}(\overline{DA})$ intersect, then

$$\square ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$$

is a quadrilateral.

Theorem Given a quadrilateral $\square ABCD$ in a metric geometry then $\square ABCD = \square BCDA = = \square CDAB = \square DABC = \square ADCB = \square DCBA = \square CBAD = \square BADC$. If both $\square ABCD$ and $\square ABDC$ exist, they are not equal.

1. Prove the above theorem.

Definition (sides, vertices, angles, diagonals, opposite vertices, adjacent sides, opposite sides)
In the quadrilateral $\square ABCD$, the sides are $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$; the vertices arc A, B, C, and D; the angles arc $\measuredangle ABC$, $\measuredangle BCD$, $\measuredangle CDA$, and $\measuredangle DAB$; and the diagonals are $\overline{AC}$ and $\overline{BD}$. The endpoints of a diagonal are called opposite vertices. If two sides contain a common vertex, the sides are adjacent; otherwise they are opposite. If two angles contain a common side, the angles arc adjacent; otherwise they arc opposite.

Theorem In a metric geometry, if $\square ABCD = \square PQRS$ then $\{A, B, C, D\} = \{P, Q, R, S\}$. Furthermore, if $A = P$ then $C = R$ and either $B = Q$ or $B = S$ so that the sides, angles, and diagonals of $\square ABCD$ are the same as those of $\square PQRS$.

2. Prove the above theorem.

Definition (convex quadrilateral)
A quadrilateral $\square ABCD$ in a Pasch geometry is a convex quadrilateral if each side lies entirely in a half plane determined by its opposite side.

3. Sketch two quadrilaterals in the Euclidean Plane, one of which is a convex quadrilateral and the other of which is not.

4. Sketch two quadrilaterals in the Poincaré Plane, one of which is a convex quadrilateral and the other of which is not.

Theorem In a Pasch geometry, a quadrilateral is a convex quadrilateral if and only if the vertex of each angle is contained in the interior of the opposite angle.

5. Prove the above theorem.

Theorem In a Pasch geometry, the diagonals of a convex quadrilateral intersect.

6. Prove the above theorem.

Theorem Let $\square ABCD$, be a quadrilateral in a Pasch gteometry. If $\overleftrightarrow{BC} \| \overleftrightarrow{AD}$ then $\square ABCD$ is a convex quadrilateral.

7. Prove the above theorem.

8. Prove that the quadrilateral $\square ABCD$ in a Pasch geometry is a convex quadrilateral if and only if each side does not intersect the line determined by its opposite side.

9. Give a "proper" definition of the interior of a convex quadrilateral. Then prove that the interior of a convex quadrilateral is a convex set.

10. Prove that in a Pasch geometry if the diagonals of a quadrilateral intersect then the quadrilateral is a convex quadrilateral.

"Prove" may mean "find a counterexample".

11. Prove that in a Pasch geometry at least one vertex of a quadrilateral is in the interior of the opposite angle.